

# Embedding Fractional Quantum Hall Solitons in M-theory Compactifications

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## Abstract

We engineer  $U(1)^n$  Chern-Simons type theories describing fractional quantum Hall solitons (QHS) in 1+2 dimensions from M-theory compactified on eight dimensional hyper-Kähler manifolds as target space of  $N = 4$  sigma model. Based on M-theory/Type IIA duality, the systems can be modeled by considering D6-branes wrapping intersecting Hirzebruch surfaces  $F_0$ 's arranged as *ADE* Dynkin Diagrams and interacting with higher dimensional R-R gauge fields. In the case of finite Dynkin quivers, we recover well known values of the filling factor observed experimentally including Laughlin, Haldane and Jain series.

**Keywords:** Quantum Hall Solitons, M-theory compactifications, Type IIA string, 2D  $N = 4$  sigma models, ADE geometries.

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# 1 Introduction

Recently, efforts have been devoted to study connections between the quantum theory of condensed matter physics and higher dimensional supergravity models embedded in 10D type II superstrings and 11D M-theory [1, 2, 3, 4]. In particular, the fractional Quantum Hall Effect (QHE) has been subject to some interest not only because of its experimental results, including graphene [5], but also from its connection with the recent developments in brane physics using Anti de Sitter/conformal field theory (AdS/CFT) correspondence [6] and string theory compactifications [7].

The first proposed series of the fractional quantum states was given by Laughlin and they are characterized by the filling factor  $\nu_L = \frac{1}{m}$  where  $m$  is an even integer for a boson electron and an odd integer for a fermionic electron [8, 9]. At low energy, this model can be described by a 3-dimensional U(1) Chern-Simons theory coupled to an external electromagnetic field  $\tilde{A}$  with the following effective action

$$S_{CS} = -\frac{m}{4\pi} \int_{\mathbb{R}^{1,2}} A \wedge dA + \frac{q}{2\pi} \int_{\mathbb{R}^{1,2}} \tilde{A} \wedge dA \quad (1.1)$$

where  $A$  is the dynamical gauge field and  $q$  is the charge of the electron [10]. It turns out that this system can be modeled using solitonic D-branes of type II superstrings with a NS-NS B-field [11, 12, 7, 13]. When the B-field is turned on, a non commutative geometry description can be also used [14].

Following the Susskind approach and looking for extended constructions, it is not difficult to see that the most general fractional quantum Hall systems including (1.1) is given by the following abelian effective theory

$$S \sim \frac{1}{4\pi} \int \sum_{i,j} K_{ij} A^i \wedge dA^j + 2 \sum_i q_i \tilde{A} \wedge dA^i \quad (1.2)$$

where now  $K_{ij}$  is a real, symmetric and invertible matrix ( $\det K \neq 0$ ); and  $q_i$  is a vector of charges. The apparition of the  $K_{ij}$  matrix and the  $q_i$  charge vector in this effective field action are very suggestive in the sense that, besides their Lie algebra interpretation, they also capture the property to embed gauge theory in type II superstrings and, via string dualities, in M-theory compactifications. Moreover, by integrating over the gauge fields  $A^i = dx^\mu A_\mu^i$  in the same way as in Susskind model, we get the following filling factor

$$\nu = q_i K_{ij}^{-1} q_j \quad (1.3)$$

letting understand that eq(1.3) may be also thought of as giving a *unified description* of several kinds of FQH series including Laughlin, Haldane, Jain and hierarchical ones [14, 15]. An

appropriate choice of  $K_{ij}$  and  $q_i$  leads to a particular filling factor.

In this letter we discuss Fractional Quantum Hall Solitons (QHS) described by gauge quivers in 1+2 associated with the class of Kac-Moody Lie algebras having  $\det K \neq 0$ . This class includes the subset of ordinary Lie algebras classified by Cartan and a sector in the so called indefinite subset [16]. Notice that dealing with a general form of  $K_{ij}$  would be interesting; but this requires considering Borcherds algebras [17]. However the geometric interpretation of these exotic Borcherds symmetries go beyond the usual intersecting 2-(4-) cycles as it happens generally in the embedding of quiver gauge theories in superstring compactifications.

As such in our realization, the matrix  $K_{ij}$  will be identified with the Cartan matrix and its extensions to those having  $\det K \neq 0$  with indefinite sign such as hyperbolic algebras considered in [18]. The corresponding models are obtained from a direct compactification of M-theory on a real eight dimensional manifold (complex 4-dimensional hyper-Kähler manifold). The geometry is realized explicitly as a cotangent bundle over a collection of intersecting Hirzebruch surfaces<sup>1</sup> ( $F_0$ ) arranged as Dynkin diagrams. Notice by the way that  $F_0$  is a complex compact surface with simpler homology properties; general realizations are also possible; for instance by considering del Pezzo surfaces [19] à la F-theory-GUT [20, 21] where the  $K_{ij}$  is encoded in the degeneracy of the elliptic fiber<sup>2</sup> of the elliptically fibered CY 4-folds (see also the comment made in the conclusion section). More general extensions could be also done for toric varieties in which the matrix  $K_{ij}$  is identified with Mori matrices.

Moreover, based on M-theory/Type IIA duality, we give  $N = 1$  supersymmetric  $U(1)^n$  Chern-Simons type theories describing 3-dimensional QHS using D6-branes wrapping intersecting  $F_0$ 's in the presence of higher dimensional R-R gauge fields. In the case of finite Dynkin quivers, we geometrically recover some very known values observed experimentally including the Jain's series. We expect to have similar results for hyperbolic quiver models with negative filling fractions describing holes.

The organization of this work is as follows. First, we give the M-theory background as target space of  $N = 4$  sigma model. In section 3, we derive QHS in 1+2 dimensions using M-theory/Type IIA duality. In section 4, we compute the filling factor for  $A_n$  quivers. For a particular choice of the vector charge, we can recover the Jain's series. Then we extend the analysis to  $DE$  quivers in section 5. Our conclusion and comments are given in section 6.

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<sup>1</sup>The Hirzebruch surface ( $F_0$ ) is defined by a trivial fibration of  $\mathbb{CP}^1$  over  $\mathbb{CP}^1$ ; it is a particular del Pezzo surface ( $dP_1$ ) with toric realization given by a rectangle.

<sup>2</sup>It would be interesting to deepen this issue as it includes the remarkable case  $\det K_{ij} = 0$  which is associated with affine singularities [21, 22] and also space-time conformal symmetry [18].

## 2 M-theory background

In this section we give the geometric background that we will use to derive QHS in 1+2 dimensions from M-theory compactified on a special class of eight dimensional manifold  $M_8$  namely a 4-dimensional complex hyper-Kähler manifold

$$\mathbb{R}^{1,2} \times M_8 \quad (2.1)$$

leading to a  $N = 2$  supersymmetric low energy effective Chern-Simons gauge theory in 3D space time. For a type II stringy interpretation,  $M_8$  will be viewed as a circle fibration over a real 7-dimensional base manifold  $M_7$ . Using M-theory/Type IIA duality, the base  $M_7$  can be identified with a local type IIA geometry. A nice way to describe  $M_8$  is to use the so-called hyper-Kähler quotient studied in [23] engineered by considering a two-dimensional  $U(1)^r$  sigma model with eight supercharges and  $r + 2$  hypermultiplets. There is a  $SU(r + 2)$  global symmetry under which the hypermultiplets transform in the fundamental representation  $r + 2$ . Thus,  $M_8$  is defined by the following D-flatness condition

$$\sum_{i=1}^{r+2} Q_i^a [\phi_i^\alpha \bar{\phi}_{i\beta} + \phi_i^\beta \bar{\phi}_{i\alpha}] = \vec{\xi}_a \vec{\sigma}_\beta^\alpha, \quad a = 1, \dots, r \quad (2.2)$$

where  $Q_i^a$  is a matrix charge which will be identified here with the extended Cartan matrices [16].  $\phi_i^\alpha$ 's ( $\alpha = 1, 2$ ) denote the component field doublets of each hypermultiplets ( $i = 1 \dots, r + 2$ ).  $\vec{\xi}_a$  are the Fayet-Iliopolos (FI) 3-vector couplings rotated by  $SU(2)$  symmetry, and  $\vec{\sigma}_\beta^\alpha$  are the traceless  $2 \times 2$  Pauli matrices. Performing  $SU(2)$  R-symmetry transformations  $\phi^\alpha = \varepsilon^{\alpha\beta} \phi_\beta$ ,  $\bar{\phi}^\alpha = \bar{\phi}_\alpha$ ,  $\varepsilon_{12} = \varepsilon^{21} = 1$  and replacing the Pauli matrices by their expressions, the identities (2.2) can be split as follows

$$\begin{aligned} \sum_{i=1}^{r+2} Q_i^a (|\phi_i^1|^2 - |\phi_i^2|^2) &= \xi_a^3 \\ \sum_{i=1}^{r+2} Q_i^a \phi_i^1 \bar{\phi}_i^2 &= \xi_a^1 + i \xi_a^2 \\ \sum_{i=1}^{r+2} Q_i^a \phi_i^2 \bar{\phi}_i^1 &= \xi_a^1 - i \xi_a^2 \end{aligned} \quad (2.3)$$

Up to some technical details, the general solution of these equations can be viewed as the cotangent bundle<sup>3</sup> over a collection of  $r$  intersecting  $F_0 \sim \mathbb{CP}^1 \times \mathbb{CP}^1$  arranged as Dynkin diagrams of Kac-Moody Lie algebras [24, 25]. This result, which generalizes the  $N = 2$  scenario dealing with the ALE spaces in which appear only intersecting  $\mathbb{CP}^1$ 's, leads to a

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<sup>3</sup> $F_0$  is the blow up of  $CP^2$  at a point; it is just  $dP_1$  the leading term of the del Pezzo surfaces  $dP_n$  given by the blow ups of  $CP^2$  at  $n$  points with  $n \leq 8$ . It would be interesting to extend the present construction to intersecting  $dP_n$ 's.

nice correspondence between the root lattice of Kac-Moody Lie algebras and the set of  $F_0$ 's forming a basis of the cohomology space of four-cycles  $H_4(M_8, \mathbb{Z})$ . This connection can be supported by the intersection theory of complex surfaces inside  $M_8$ . In fact, the self-intersection of the zero section in the cotangent bundle of  $F_0$  is equal to its minus Euler number, i.e.  $-4$ . Assuming that  $F_0^i$  intersects  $F_0^{i+1}$  at two points, which can be supported by the fact that each  $\mathbb{CP}^1$  inside  $F_0^i$  intersects just one  $\mathbb{CP}^1$  in the next  $F_0^{i+1}$ , we get the intersection numbers of the  $F_0$ 's

$$[F_0^i] \cdot [F_0^i] = -4, \quad [F_0^i] \cdot [F_0^{i+1}] = 2, \quad (2.4)$$

with others vanishing. This means that  $F_0^i$  does not intersect  $F_0^j$  if  $|j - i| > 1$ . Up to a multiplication factor by two, the intersection numbers reproduce the elements of the Cartan matrices  $C_{ij}$ . The compact intersecting geometry agrees with the extended Dynkin diagrams. More specifically, with each simple root  $\alpha_i$ , we associate a single  $F_0^i$  and we have the following intersection form

$$[F_0^i] \cdot [F_0^j] = -2C^{ij}. \quad (2.5)$$

It should be interesting to write down the algebraic equations dealing with the corresponding singularities extending the case of ALE spaces [26]. We expect that this could be obtained in terms of  $N = 4$  sigma model gauge invariants.

### 3 QHS from M-theory/Type IIA duality

Having specified the geometric background, we will give a M-theory QHS in terms of effective Chern-Simons type theory with a series of  $U(1)$  gauge fields. This internal space presents *ADE* quiver description of QHS in 1+2 dimensions which are obtained from a direct M-theory compactification instead of type IIA moving on  $ALE \times S^3$  and performing a Kaluza-Klein compactification on  $S^3$  [12]. Our way provides a different brane system dual to M-theory geometric background leading to fractional QHS. Besides that, it will allow us to derive directly some filling fractions which coincide with a subsequence of the celebrated Jain's series. This can be recovered as exact values without modifying the intersection matrix associated with the re-normalization of the inner product between simple roots as made in [27] for the *ALE* space in the presence of D4-branes. Moreover, the M-theory internal space can develop singularities of many different types and we expect that instead of *ADE* singularities, one could in a similar way analyze other intersecting geometries. This may lead to new quiver models based on resolving such singularities.

Roughly, our analysis will be based on a dual type IIA local geometry in presence of D6-branes interacting with R-R higher dimensional gauge fields. By the use of M-theory/Type IIA duality, it is worth noting that M-theory on  $M_8$  is expected to be dual to Type IIA super-

string on  $M_8/U(1)$ , where  $U(1)$  can be identified with the M-theory circle compactification (going from eleven dimensions to ten). A priori, there are many ways one may follow to choose the  $U(1)$  symmetry. Here we will use the circle actions involved in toric geometry which is a powerful tool for studying complex manifolds in terms of simple combinatorial data of polytopes [28]. The simple example of toric varieties is the complex plane  $\mathbb{C}$ . The latter admits an  $U(1)$  toric action

$$z \rightarrow ze^{i\theta}, \quad (3.1)$$

which has a fixed point at  $z = 0$ . The toric geometry of  $\mathbb{C}$  can be viewed as a circle fibred on a half line parameterized by  $|z|$ . The circle determined by the action of  $\theta$  shrinks at  $z = 0$ . This realization can be generalized easily to  $\mathbb{C}^n$  space where we have a  $T^n$  fibration, parameterized by the angular coordinates  $\theta_i$ , over a  $n$ -dimensional real base parameterized by  $|z_i^2|$ . The more interesting compact example in toric geometry is the  $\mathbb{CP}^1$  space admitting an  $U(1)$  toric action having two fixed points describing respectively the north and the south poles of the two sphere  $S^2 \sim \mathbb{CP}^1$ . In this way,  $\mathbb{CP}^1$  can be viewed as an interval fibred by  $S^1$  with zero size at the two boundaries. Using these ideas, our geometry  $M_8$  can be viewed also as a toric space admitting four toric geometry circle actions  $U(1)_{base}^2 \times U(1)_{fiber}^2$ ; two of them correspond to the  $F_0$ 's base space denoted by  $U(1)_{base}^2$  while the remaining ones  $U(1)_{fiber}^2$  act on the fiber cotangent directions. Dividing by one finite fiber circle action

$$M_7 = \frac{M_8}{U(1)_{fiber}}, \quad (3.2)$$

we can obtain a 7-dimensional type IIA geometry. For instance, in the case of two dimensional  $U(1)^r$  sigma model with the finite  $A_r$  Cartan matrix gauge charges and  $r + 2$  hypermultiplets, this quotient space becomes a real cone on a  $S^2$  bundle over a collection of  $r$  intersecting  $F_0$ 's arranged as Dynkin diagram of  $A_r$  finite Lie algebras, preserving  $N = 2$  supersymmetry in  $2 + 1$  dimensions.

In the following, we will show that the dual type IIA geometry can generate  $U(1)^n$  Chern-Simons type theories from D6-branes wrapped on intersecting  $F_0^i$ 's and filling the 3-dimensional Minkowski space on which QHS will reside. To obtain an effective theory of hierarchical description QHS, let us first start with a Chern-Simons theory with a single  $U(1)$  gauge symmetry. The corresponding geometry can be obtained from a sigma model with one  $U(1)$  gauge symmetry and  $A_1$  vector charge. In this case, the local type IIA geometry reduces to a real cone over

$$S^2 \times F_0 \quad (3.3)$$

Using arguments similar to [12, 29], we can wrap a D6-brane over the zeroth Hirzebruch surface  $F_0$  to get the Chern-Simons action (1.1). Indeed, on the seven-dimensional world-

volume of each D6-brane we have U(1) gauge symmetry. The corresponding effective theory has two parts:

$$S_{D6} = S_{DBI} + S_{WZ}. \quad (3.4)$$

The DBI part is given by

$$S_{DBI} \sim T_6 \int d^7\sigma e^{-\phi} \sqrt{-\det(G + 2\pi F)} \quad (3.5)$$

while the WZ action reads as

$$S_{WZ} \sim T_6 \int_{\mathbb{R}^{1,2} \times F_0} F \wedge F \wedge C_3 \quad (3.6)$$

where  $T_6$  is the brane tension and where  $C_3$  is the R-R 3-form coupled to the D2-brane of type IIA superstring. Ignoring the first terms and integrating by part, the WZ action on the D6-brane world-volume becomes

$$\int_{\mathbb{R}^{1,2} \times F_0} F \wedge F \wedge C_3 = - \int_{\mathbb{R}^{1,2} \times F_0} A \wedge F \wedge (dC)_4 \quad (3.7)$$

Now, integrating over  $F_0$ , we get the first Chern-Simons terms

$$- \frac{m}{4\pi} \int_{\mathbb{R}^{1,2}} A \wedge F \quad (3.8)$$

where  $m = \frac{1}{2\pi} \int_{F_0} (dC)_4$  which can be computed from intersection theory of  $M_8$ . To couple the system to an external gauge field, we need to introduce the RR 5-form  $C_5$  which is sourced by a D4-brane. This gauge field decomposes as follows

$$C_5 \rightarrow \tilde{A} \wedge \omega \quad (3.9)$$

where  $\omega$  is a harmonic 4-form on  $F_0$ . In this way, the WZ term  $\int C_5 \wedge F$  on a D6-brane gives

$$q \int_{\mathbb{R}^{1,2}} \tilde{A} \wedge F \quad (3.10)$$

where  $\tilde{A}$  is the U(1) gauge field which can be obtained from the dimensional reduction of the RR 5-form on  $F_0$ . This U(1) gauge field can be interpreted as a magnetic external gauge field that couples to our QHS. We can follow the same steps to construct an effective Chern-Simons gauge theory with a series of U(1) gauge fields which is called hierarchical description. The corresponding effective action can be obtained from a stack of D6-branes wrapping individually intersecting  $F_0$ 's. Using the intersection contributions, the action can take the same form as in (1.2) where now  $K_{ij}$  can be identified with the intersection matrix

of  $F_0$ 's (4.2), and where  $q_i$  is the vector field characterizing a fractional quantum Hall theory associated with  $ADE$  geometries. Since  $K_{ij}$  and  $q_i$ , which are interpreted as order parameters classifying the various FQH states, are related to the filling factor [30], equation (1.3) can be solved algebraically in terms of representation theory of the Kac-Moody Lie algebras [27].

## 4 ADE quiver models

In this section, we present concrete examples of the more formal results that we have developed above for which Chern-Simons gauge theory provides an  $ADE$  effective field description. For simplicity, we will mainly consider the  $ADE$  finite quiver gauge models by introducing 4-cycles in the base which are intersecting according to  $ADE$  Dynkin graph. We refer to these models as  $ADE$  quiver models. They constitute a very natural class of models in this context. It is pointed out, though, that we in principle could consider more complicated geometries. We will restrict ourselves to the simply laced  $ADE$  ones as they allow us to extract the corresponding physics in a straightforward manner. In this case, the quadratic form (1.3) can be written as

$$\nu = \frac{1}{2} \sum_i C_{ii}^{-1} q_i^2 + \sum_{i < j} C_{ij}^{-1} q_i q_j. \quad (4.1)$$

### 4.1 $A_n$ quiver models

As an illustration, we now consider the  $A_n$  quivers corresponding to  $U(1)^n$  quiver gauge theory based on finite  $A_n$  Dynkin diagram. In the context of Type IIA string theory, this model appears as the world-volume of  $n$  D6-branes wrapping separately  $F_0$ 's arranged as follows

$$A_n : \quad \bigcirc \text{---} \bigcirc \text{---} \bigcirc \cdots \cdots \bigcirc \text{---} \bigcirc \quad (4.2)$$

For a generic vector charge, the filling factor (4.1) is given now by the following quadratic form

$$\nu(A_n) = \frac{1}{n+1} \left( \frac{1}{2} \sum_{i=1}^n i(n-i+1) q_i^2 + \sum_{i < j} i(n-j+1) q_i q_j \right). \quad (4.3)$$

In the quantum Hall literature, the simplest model is related to single layer FQH states. In this case, the components of the vector charge are zeros except one entry which is equal 1. When taking into account that  $q_i = \delta_{i,p}$  for some  $p = 1, \dots, n$ , we obtain the relation

$$\nu(A_n) = \nu_{p,n} = \frac{p(n+1-p)}{2(n+1)}, \quad (4.4)$$

admitting the obvious symmetry  $\nu_{p,n} = \nu_{n+1-p,n}$ . Actually, this can be viewed as a generalization of the result given in [7]. To see that, specializing the computation to  $p = 1$  (or  $n$ ), we get  $\nu_{1,n} = \frac{n}{2(n+1)}$ . Taking  $n = 2m$  (even number of  $F_0$ 's), the filling factor can be written as

$$\nu_{1,2m} = \frac{m}{2m+1}, \quad (4.5)$$

which interestingly coincides with a subsequence of the Jain's series given by

$$\nu_{\text{Jain}} = \frac{m}{mk \pm 1} \quad m, k/2 = 1, 2, 3, \dots \quad (4.6)$$

This contains some experimentally observed filling fractions. It has been shown that the hierarchy scheme of states proposed by Jain [31] for the fractional quantum Hall effect can be viewed in terms of an effective theory of composite fermions [32]. There the electrons are thought of as dressed by magnetic fluxes leading to the filling factor. In type IIB superstring picture, it has been shown that fractional quantum Hall states for the Jain filling factor can be related with the integer quantum Hall states of the composite fermions. Using an effective field theory description with Chern-Simons action, the Jain relation can be interpreted as the result of the perturbative renormalization of the integer QHE by the auxiliary heavy fermions. More details on this construction are given in [33]. However, here our analysis is based on hierarchy descriptions using Chern-Simons models dual to M-theory compactification on  $ADE$  eight dimensional manifolds. It should be interesting to find the connection between these two ways using string theory data. This will be addressed elsewhere.

In the end of this subsection, we note that the integer  $m$  appearing in the Jain's series is related to the dimension of  $H_4(M_8, \mathbb{Z})$  for the  $A_{2m}$  geometry. At this level, one may ask the following question: Is there any interpretation for the integer  $k$ ? In what follows, we speculate on it. For this reason, it may be useful to introduce geometries with genus  $g > 0$ . In fact, we will replace the zeroth Hirzebruch  $F_0$  surface by  $F_{0,g} = \Sigma_g \times \Sigma_g$  where  $\Sigma_g$  is the Riemann surface of genus  $g$  that substitutes the  $\mathbb{CP}^1$  sphere. Suppose, for simplicity, that the cohomology space  $H_4(M_8, \mathbb{Z})$  is of rank 2 and that  $F_{0,g}^1$  and  $F_{0,g}^2$  are complex surfaces representing its generators  $[F_{0,g}^1], [F_{0,g}^2]$ . The intersection form is given by

$$\begin{pmatrix} [F_{0,g}^1] \cdot [F_{0,g}^1] & [F_{0,g}^1] \cdot [F_{0,g}^2] \\ [F_{0,g}^2] \cdot [F_{0,g}^1] & [F_{0,g}^2] \cdot [F_{0,g}^2] \end{pmatrix}, \quad (4.7)$$

where now  $[F_{0,g}^1] \cdot [F_{0,g}^2]$  denotes the algebraic intersection of  $F_{0,g}^1$  and  $F_{0,g}^2$ . For any manifold  $F_{0,g}$ , the geometry can be regarded as the zero section of the cotangent bundle over  $F_{0,g}$ . Thus,

the self intersection is equal to its minus Euler number

$$[F_{0,g}] \cdot [F_{0,g}] = 2\Sigma_g \cdot \Sigma_g = 2(2g - 2).$$

To get the intersection number between  $F_{0,g}^1$  and  $F_{0,g}^2$ , one may extend the result of the Riemann surfaces with genus 0 and 1. Assuming that the complex surfaces meet negatively for  $g > 1$ , we expect to have the following intersection form

$$[F_{0,g}^1] \cdot [F_{0,g}^2] = 2(1 - g) \quad (4.8)$$

For more general extended geometries, one expects to have

$$[F_{0,g}^i] \cdot [F_{0,g}^j] = 2(g - 1)C^{ij}. \quad (4.9)$$

Now, specializing the computation to  $A_{2(g-1)n}$ , for  $g > 1$ , quiver gauge theory for the vector charge  $q_i = \delta_{i,p}$  with  $p = 1$  or  $n$ , we get a filling factor coincides exactly with a subsequence of Jain's series and the integer  $k$  can be expressed in terms of the genus as follows

$$k = 2(g - 1). \quad (4.10)$$

which is, as it should be, even integer.

## 4.2 DE quiver gauge models

We have mainly considered finite  $A_n$  quivers. Here we would like to generalize this to the case where the quiver gauge models are based on finite *DE* Dynkin diagrams. The general structure is quite like what we have seen for  $A_n$ . First of all, in QHE, it is very crucial that the filling fraction has odd-denominator (actually, there is no intrinsically string theoretic understanding of the opposite). Admitting this fact, we see that  $D_n$  and  $E_7$  quiver gauge models do not provide "expected" results because the corresponding Cartan matrices have even-determinant (4 and 2 respectively). As for the algebra  $E_8$ , for which the determinant is equal to 1, it always gives whole filling factors. However, in the case of  $U(1)^6$  quiver gauge models based on finite  $E_6$  Dynkin graph, we get interesting fractions. For further illustration we display, in the table below, our calculations of the filling factor  $\nu$  concerning the *DE* quiver

gauge models for typical values of the vector charge

|       | vector charge          | $\nu$ |
|-------|------------------------|-------|
| $D_n$ | (10...00)              | 1/2   |
|       | (00...01) or (0...010) | $n/8$ |
| $E_6$ | (100000) or (000010)   | 2/3   |
| $E_7$ | (0000010)              | 3/4   |
| $E_8$ | (00000010)             | 1     |

(4.11)

The value  $\nu = 2/3$  associated with the  $E_6$  case is more convincing one. In fact, it belongs to the “minus” subsequence of the Jain’s series:  $\nu = \frac{n}{2n-1}$  with  $n = 2$ .

## 5 Conclusion and comments

In this letter, we have given M-theory derivation of FQHS using 3-dimensional Chern-Simons gauge theories based on Dynkin diagrams. This construction, based on Lie algebras, leads to a general form of the filling factor (1.3) and gives a unified description of several kinds of FQH series including Laughlin, Haldane, Jain and hierarchical ones.

Using M-theory/Type IIA duality, we have reproduced fractional values of the filling factor observed experimentally using D6-branes wrapping a collection of intersecting  $F_0$  geometries according to Dynkin diagrams of finite  $A_n$  type algebras. In particular, the Jain’s series can be recovered by considering quiver gauge theory based on  $A_{2m}$  Dynkin diagrams. However, for  $n = 2m + 1$  we get  $\nu = 1 - \frac{1}{m+1}$  which coincides with known filling fractions in the literature [10]. We have also analyzed the finite  $DE$  quiver gauge theories.

Our approach is adaptable to a broad variety of geometries whose intersection forms may be represented by extended Cartan matrices. We intend to discuss elsewhere the extension of this explicit study to the indefinite extended geometries as well as those brane realizations based on D6-branes wrapping del Pezzos  $dP_n$  with  $1 < n \leq 8$  where  $F_0$  is just the leading  $dP_1$ . In connection with that, it would therefore be of interest to consider the two following:

(1) use the tools developed recently in the framewok of F-theory-GUT to construct new classes of QHS that are embedded in M-theory on CY4-folds. Obviously, the physics between the two topics is different; the role played by the 7-brane wrapping del Pezzos in F-theory GUT is now played by 6-brane wrapping the same 4-cycles. In other words, in both F-theory-GUT and QHS in M-theory, we have the same kind of CY4-folds.

(2) choose a given FQHE series  $\nu$ , and solve eq(1.3) to end with a matrix which is not nec-

essary of Cartan type. We then expect the existence of models that are associated with Borchers symmetries giving one more evidence for these kinds of M-theory symmetries. For a connection between Borchers and M-theory, see [34] and refs therein. We believe that these issues deserve to be studied further.

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